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Rainbow connection number and the number of blocks*

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Abstract

An edge-colored graph G is *rainbow connected* if every pair of vertices of G are connected by a path whose edges have distinct colors. The *rainbow connection number* $rc(G)$ of G is defined to be the minimum integer t such that there exists an edge-coloring of G with t colors that makes G rainbow connected. For a graph G without any cut vertex, i.e., a 2-connected graph, of order n, it was proved that $rc(G) \leq \lceil \frac{n}{2} \rceil$ and the bound is tight. In this paper, we prove that for a connected graph G of order n with cut vertices, $rc(G) \leq \frac{n+r-1}{2}$, where r is the number of blocks of G with even orders, and the upper bound is tight. Moreover, we also obtain a tight upper bound for a bridgeless graph, i.e., a 2-edge-connected graph.

Keywords: rainbow edge-coloring, rainbow connection number, cut vertex, block decomposition.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. For notation and terminology not defined here, we refer to [2]. In an edge-colored graph G, a path is called a *rainbow path* if the colors of its edges are distinct. The graph G is called *rainbow connected* if every pair of vertices are connected by at least one rainbow path in G. An edge-coloring of a connected graph G that makes G rainbow connected is called *a rainbow edge-coloring* of G. The minimum number of colors required to rainbow color G is called the *rainbow connection number* of G, denoted by $rc(G)$. If a graph G has an edge-coloring c and G' is

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a subgraph of G, $c(G')$ denotes the set of colors appeared in G'. An edge-coloring using k colors is addressed as a k-edge-coloring. If P is a path, the length of P is denoted by $\ell(P)$.

Let G' be a subgraph of a graph G. An *ear* of G' in G is a nontrivial path in G whose end vertices lie in G' but whose internal vertices are not. An *ear decomposition* of a 2-connected graph G is a sequence of subgraphs $G_0, G_1, \cdots, G_k$ of G satisfying that (1) G_0 is a cycle of G; (2) $G_i = G_{i-1} \bigcup P_i$ $(1 \le i \le k)$, where P_i is an ear of G_{i-1} in G; (3) $G_{i-1}(1 \le i \le k)$ is a proper subgraph of G_i; (4) $G_k = G$. If $\ell(P_1) \ge \ell(P_2) \ge \cdots \ge \ell(P_k)$, we say that the ear decomposition is nonincreasing. From the above definition, every graph G_i in an ear decomposition is 2-connected.

A *block* of a graph G is a maximal connected subgraph of G that does not have any cut vertex. So every block of a nontrivial connected graph is either a K_2 or a 2-connected subgraph. All the blocks of a graph G form a *block decomposition* of G. A block B is called an *even (odd) block* if the order of B is even (odd).

Let c be a rainbow k-edge-coloring of a connected graph G. If a rainbow path P in G has length k, we call P *a complete rainbow path*; otherwise, it is *an incomplete rainbow path*. A rainbow edge-coloring c of G is *incomplete* if for any vertex $u \in V(G)$, G has at most one vertex v such that all the rainbow paths between u and v are complete; otherwise, it is *complete*.

The definition of a rainbow coloring was introduced by Chartrand et al. in [5]. For more knowledge, we refer to [10, 11]. In [6], it was proved that computing the rainbow connection number of a graph is NP-hard. Hence, tight upper bounds of the rainbow connection number for a connected graph have been a subject of investigation. The authors of [4] proved that $rc(G) \le 3n/(\delta + 1) + 3$, where δ is the minimum degree of the connected graph G, and the authors of [1, 7] obtained some upper bound of the rainbow connection number in term of radius and bridges. For 2-connected graphs, there exist the following results.

Lemma 1.1. *[9] Let G be a Hamiltonian graph of order n $(n \ge 3)$. Then G has an incomplete $\lceil \frac{n}{2} \rceil$-rainbow coloring, i.e., $rc(G) \le \lceil \frac{n}{2} \rceil$.*

Lemma 1.2. *[9] Let G be a 2-connected non-Hamiltonian graph of order n $(n \ge 4)$. If G has at most one ear with length 2 in a nonincreasing ear decomposition, then G has a incomplete $\lceil \frac{n}{2} \rceil$-rainbow coloring, i.e., $rc(G) \le \lceil \frac{n}{2} \rceil$.*

Theorem 1.1. *[9, 8] Let G be a 2-connected graph of order n $(n \ge 3)$. Then $rc(G) \le \lceil \frac{n}{2} \rceil$, and the upper bound is tight for $n \ge 4$.*

Proposition 1.1. *[3] If G is a connected bridgeless (2-edge-connected) graph with n vertices, then $rc(G) \le 4n/5 - 1$.*

In this paper, we will study the rainbow connection number of a connected graph with cut vertices and obtain a tight upper bound. Besides, a tight upper bound for a 2-edge-connected (bridgeless) graph is also obtained.

2 Main results

We first show that every 2-connected graph G with odd number of vertices has a rainbow edge-coloring with a nice property.

Lemma 2.1. *Let G be a 2-connected graph of order n ($n \geq 3$) and v_0 be any vertex of G. If n is odd, then G has a rainbow $\lceil \frac{n}{2} \rceil$-edge-coloring c such that there exists a color x of the edge-coloring satisfying that every vertex of G can be connected by a rainbow path P to v_0 with $x \notin c(P)$.*

Proof. Since G is 2-connected, G has a nonincreasing ear decomposition $G_0, G_1, \cdots, G_q(= G)$ ($q \geq 0$) satisfying that (1) G_0 is a cycle with $v_0 \in V(G_0)$; (2) $G_i = G_{i-1} \bigcup P_i$, where P_i ($1 \leq i \leq q$) is an ear of G_{i-1} in G; (3) $\ell(P_1) \geq \ell(P_2) \geq \cdots \geq \ell(P_q)$. We consider the following two cases.

Case 1. No ear of $P_1, \cdots, P_q$ has an even length.

In this case, since G has an odd order, G_0 must be an odd cycle. Assume that $G_0 = v_0v_1 \cdots v_{2k}v_{2k+1}(= v_0)$ with $k \geq 1$. Define a $(k+1)$-edge-coloring c_0 of G_0 by $c_0(v_{i-1}v_i) = x_i$ for i with $1 \leq i \leq k+1$ and $c_0(v_{i-1}v_i) = x_{i-k-1}$ for i with $k+2 \leq i \leq 2k+1$. It can be checked that c_0 is a rainbow $\lceil \frac{|V(G_0)|}{2} \rceil$-edge-coloring of G_0 such that every vertex of G_0 can be connected by a rainbow path P in G_0 to v_0 with $x_{k+1} \notin c_0(P)$. If $G_0 = G$, the conclusion holds.

Now assume that $G_0 \neq G$ and $P_1 = v'_0v'_1 \cdots v'_{2s}v'_{2s+1}(s \geq 0)$ with $V(G_0) \bigcap V(P_1) = \{v'_0, v'_{2s+1}\}$. Define an edge-coloring c_1 of $G_1 = G_0 \bigcup P_1$ by $c_1(e) = c_0(e)$ for $e \in E(G_0)$, $c_1(v'_{i-1}v'_i) = y_i$ for i with $1 \leq i \leq s$, $c_1(v'_sv'_{s+1}) = x'$ and $c_1(v'_{i-1}v'_i) = y_{i-s-1}$ for i with $s+2 \leq i \leq 2s+1$, where $y_1, \cdots, y_s$ are new colors and x' is a color that already appeared in G_0. Here, if $\ell(P_1) = 1$, i.e., $s = 0$, we just color the only edge $v'_0v'_1$ of P_1 by a color that appeared in G_0. It can be checked that c_1 is a rainbow $\lceil \frac{|V(G_1)|}{2} \rceil$-edge-coloring of G_1. From the definition of c_1, every vertex of G_0 can be connected by a rainbow path P in G_0 to v_0 with $x_{k+1} \notin c_1(P)$. Let P' and P'' be the rainbow paths, respectively, from v'_0 and v'_{2s+1} to v_0 in G_0 such that $x_{k+1} \notin c_1(P')$ and $x_{k+1} \notin c_1(P'')$. For any vertex v'_j ($1 \leq j \leq s$), $v'_jP_1v'_0P'v_0$ is a rainbow path in G_1 from v'_j to v_0 such that $x_{k+1} \notin c_1(v'_jP_1v'_0P'v_0)$. For any vertex v'_j ($s+1 \leq j \leq 2s$), we can choose $v'_jP_1v'_{2s+1}P''v_0$ as a rainbow path in G_1 from v'_j to v_0 such that $x_{k+1} \notin c_1(v'_jP_1v'_{2s+1}P''v_0)$. Hence, c_1 is a required rainbow edge-coloring of G_1.

If $G_1 = G$, the conclusion holds. Otherwise, repeating the above process of defining c_1 from c_0, we can obtain a rainbow $\lceil \frac{|V(G_i)|}{2} \rceil$-edge-coloring of G_i ($2 \leq i \leq q$) such that every vertex of G_i can be connected by a rainbow path P in G_i to v_0 with $x_{k+1} \notin c_i(P)$. Therefore, c_q is a required rainbow $\lceil \frac{n}{2} \rceil$-edge-coloring of G.

Case 2. At least one of $P_1, \cdots, P_q$ has an even length.

Suppose that P_t ($1 \leq t \leq q$) is the last added ear with an even length. So $P_{t+1}, \cdots, P_s$ have odd lengths. From Case 1, we just need to show that G_t has a required rainbow

$\lceil \frac{|V(G_t)|}{2} \rceil$-edge-coloring. Now we will consider the following two cases:

Subcase 2.1. At most one of the ears $P_1, \cdots, P_{t-1}$ has length 2.

Assume that $P_t = v'_0 v'_1 \cdots v'_{2s-1} v'_{2s}$ such that $V(P_t) \bigcap V(G_{t-1}) = \{v'_0, v'_{2s}\}$. It is obvious that $G_0, G_1, \cdots, G_{t-1}$ is a nonincreasing ear decomposition of G_{t-1} with at most one ear with length 2. From Lemmas 1.1 and 1.2, G_{t-1} has an incomplete rainbow $\lceil \frac{|V(G_{t-1})|}{2} \rceil$-edge-coloring c_{t-1}. In G_{t-1}, there exists an incomplete rainbow path P' from v_0 to one of v'_0 and v'_{2s} (say v'_{2s}). Assume that x' is a color of the coloring c_{t-1} with $x' \notin c_{t-1}(P')$. Define an edge-coloring c_t of $G_t = G_{t-1} \bigcup P_t$ by $c_t(e) = c_{t-1}(e)$ for $e \in E(G_{t-1})$, $c_t(v'_{i-1}v'_i) = x_i$ for i with $1 \le i \le s$, $c_t(v'_s v'_{s+1}) = x'$ and $c_t(v'_{i-1}v'_i) = x_{i-s-1}$ for i with $s+2 \le i \le 2s$, where $x_1, \cdots, x_s$ are new colors. It can be checked that c_t is a rainbow $\lceil \frac{|V(G_t)|}{2} \rceil$-edge-coloring of G_t. From the definition of coloring c_t, every vertex of G_{t-1} has a rainbow path P in G_{t-1} to v_0 with $x_s \notin c_t(P)$. Let P'' be a rainbow path in G_{t-1} from v'_0 to v_0. For any vertex v'_j $(1 \le j \le s-1)$, $v'_j P_t v'_0 P'' v_0$ is a rainbow path in G_t from v'_j to v_0 such that $x_s \notin c_t(v'_j P_t v'_0 P'' v_0)$. For any vertex v'_j $(s \le j \le 2s-1)$, we have $v'_j P_t v'_{2s} P' v_0$ is a rainbow path in G_t from v'_j to v_0 such that $x_s \notin c_t(v'_j P_t v'_{2s} P' v_0)$. So every vertex of G_t has a rainbow path P in G_t to v_0 with $x_s \notin c_t(P)$. Hence, c_t is a required rainbow edge-coloring of G_t.

Subcase 2.2. At least two ears of $P_1, \cdots, P_{t-1}$ have length 2.

In this case, it is obvious that $\ell(P_t) = 2$ and $\ell(P_{t+1}) = \cdots = \ell(P_q) = 1$. Assume that $\ell(P_1) \ge \cdots \ge \ell(P_h) \ge 3$ and $\ell(P_{h+1}) = \cdots = \ell(P_t) = 2$. Here at least three ears have length 2, i.e., $t - h \ge 3$. From Theorem 1.1, G_h has a rainbow $\lceil \frac{|V(G_h)|}{2} \rceil$-edge-coloring c_h. Assume that $P_j = a_j v_j b_j$ $(h+1 \le j \le t)$ such that $V(P_j) \bigcap V(G_h) = \{a_j, b_j\}$. Define an edge-coloring c_t of G_t by $c_t(e) = c_h(e)$ for $e \in E(G_h)$, $c_t(a_j v_j) = x_1$ for j with $h+1 \le j \le t$ and $c_t(v_j b_j) = x_2$ for j with $h+1 \le j \le t$, where x_1, x_2 are new colors. It is easy to check that c_t is a rainbow edge-coloring of G_t with at most $\lceil \frac{|V(G_t)|}{2} \rceil$ colors and every vertex of G_t has a rainbow path P to v_0 with $x_2 \notin c_t(P)$. Therefore, G_t has a required rainbow $\lceil \frac{|V(G_t)|}{2} \rceil$-edge-coloring. $\square$

Theorem 2.1. *Let G be a connected graph of order n $(n \ge 3)$ and G has a block decomposition $B_1, \cdots, B_q$ $(q \ge 2)$, where r blocks are even blocks and the others are odd ones. Then $rc(G) \le \frac{n+r-1}{2}$ and the upper bound is tight.*

Proof. Let G be a connected graph of order n with q $(q \ge 2)$ blocks in its block decomposition. If G has at least one even block, we choose $G_1 = B_1$ being an even block of G; otherwise, $G_1 = B_1$ being an odd block of G. Since $q \ge 2$ and G is connected, G has a block B_2 such that $V(G_1) \bigcap V(B_2) = \{v_1\}$. Let $G_2 = G_1 \bigcup B_2$. So G_2 is a connected graph which consists of two blocks B_1, B_2. Repeating the process of adding B_2 to G_1, we obtain a sequence of subgraphs $G_1, G_2, \cdots, G_q$ such that G_i $(1 \le i \le q)$ is a connected graph and $G_i = B_1 \bigcup B_2 \bigcup \cdots \bigcup B_i$ $(2 \le i \le q)$ with $V(G_{i-1}) \bigcap V(B_i) = \{v_{i-1}\}$ for i with $2 \le i \le q$. Denote the order of B_i $(1 \le i \le q)$ by n_i. From Theorem 1.1 and $rc(K_2) = 1$, every block B has a rainbow $\lceil \frac{|V(B)|}{2} \rceil$-edge-coloring. We will consider the following two

cases.

Case 1. $r \geq 1$.

From the definition of G_1, $G_1 = B_1$ is an even block and G_1 has a rainbow $\lfloor \frac{n_1}{2} \rfloor$-edge-coloring c_1. If B_2 is an even block, color the edges of B_2 with $\lfloor \frac{n_2}{2} \rfloor$ new colors such that B_2 is rainbow connected. It is obvious that G_2 is rainbow connected and the obtained edge-coloring c_2 of G_2 uses $\lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor$ colors. Consider the case that B_2 is an odd block. From Lemma 2.1, B_2 has a rainbow edge-coloring c'_2 with $\lceil \frac{n_2}{2} \rceil$ new colors such that there exists a color x' of c'_2 satisfying that every vertex of B_2 has a rainbow path P in B_2 to v_1 with $x' \notin c'_2(P)$. Replacing the color x' of c'_2 by a color x that already appeared in G_1, we obtain an edge-coloring c_2 of G_2 with $\lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor$ colors. It is obvious that G_1 and B_2 are rainbow connected, respectively. Consider two vertices $v' \in V(G_1)$ and $v'' \in V(B_2)$. From the definition of c_2, there are two rainbow paths P' in G_1 from v' to v_1 and P'' in B_2 from v_1 to v'' such that $x \notin c_2(P'')$. So $v'P'v_1P''v''$ is a rainbow path from v' to v'' in G_2. Hence, c_2 is a rainbow edge-coloring of G_2 with $\lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor$ colors.

If $q \geq 3$, we can repeat the process of defining c_2 from c_1 to obtain a rainbow edge-coloring c_q of $G_q(= G)$ with $\lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor + \cdots + \lfloor \frac{n_q}{2} \rfloor$ colors.

Case 2. $r = 0$.

In this case, $G_2 = B_1 \bigcup B_2$ consists of two odd blocks. From Lemma 2.1, B_i $(i = 1, 2)$ has a rainbow $\lceil \frac{n_i}{2} \rceil$-edge-coloring c'_i such that x'_i is a color of c'_i satisfying that every vertex of B_i $(i = 1, 2)$ has a rainbow path P in B_i to v_1 with $x'_i \notin c'_i(P)$. Note that $c'_1(B_1) \bigcap c'_2(B_2) = \emptyset$. Assume that x_i $(i = 1, 2)$ is a color of c'_i such that $x_i \neq x'_i$. Replacing x'_1 by x_2 in B_1 and x'_2 by x_1 in B_2, we obtain an edge-coloring c_2 of G_2 with $\lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor$ colors. It is obvious that B_i $(i = 1, 2)$ is rainbow connected. Consider two vertices $v' \in V(B_1)$ and $v'' \in V(B_2)$. From the definition of c_2, there exist two rainbow paths P' in B_1 from v' to v_1 and P'' in B_2 from v_1 to v'' such that $x_2 \notin c_2(P')$ and $x_1 \notin c_2(P'')$. So $v'P'v_1P''v''$ is a rainbow path in G_2 from v' to v''. Hence, c_2 is a rainbow edge-coloring of G_2 with $\lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor$ colors. If $q \geq 3$, we can color the blocks $B_3, \cdots, B_q$ similar to Case 1 to obtain a rainbow edge-coloring of G with $\lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor + \cdots + \lfloor \frac{n_q}{2} \rfloor$ colors.

Therefore, in any case we have that $rc(G) \leq \lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor + \cdots + \lfloor \frac{n_q}{2} \rfloor = \frac{n+r-1}{2}$.

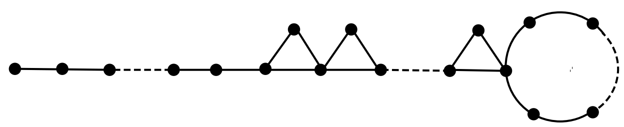

Figure 1. Graphs for the tightness of Theorem 2.1.

In order to prove that the upper bound is tight, we will show that for any integers n, r, q, if there exist graphs of order n with r even blocks and $q - r$ odd blocks, then one of such graphs has a rainbow connection number $\frac{n+r-1}{2}$.

In fact, if there exists a connected graph of order n with r even blocks, then $n + r$

must be an odd number. The graph G of order n in Figure 1 consists of r even blocks K_2, $q-r-1$ odd cycles K_3 and one odd cycle $C_{n-2q+r+2}$. Since $d(G) = \frac{n+r-1}{2}$ and $d(G) \leq rc(G) \leq \frac{n+r-1}{2}$, we have $rc(G) = \frac{n+r-1}{2}$. $\square$

In the following, we give a tight upper bound of the rainbow connection number for a 2-edge-connected graph which improves the result of Proposition 1.1.

Theorem 2.2. *Let G be a 2-edge-connected graph of order n ($n \geq 3$). Then*

$$rc(G) \leq \begin{cases} 2k & \text{if } n = 3k+1 \text{ or } 3k+2 \\ 2k+1 & \text{if } n = 3k+3 \end{cases},$$

and the upper bound is tight.

Proof. Suppose that G has the block decomposition $B_1, B_2, \cdots, B_q$. Since G is 2-edge-connected, we have $|B_i| \geq 3, 1 \leq i \leq q$. And if B_i is an even block, then $|B_i| \geq 4$. If G has r even blocks, then $3r+1 \leq n$, i.e., $r \leq \frac{n-1}{3}$. From Theorem 2.1, $rc(G) \leq \frac{n+r-1}{2} \leq \frac{2n-2}{3}$. Since $rc(G)$ is an integer, $rc(G) \leq \begin{cases} 2k & \text{if } n = 3k+1 \text{ or } 3k+2 \\ 2k+1 & \text{if } n = 3k+3 \end{cases}$.

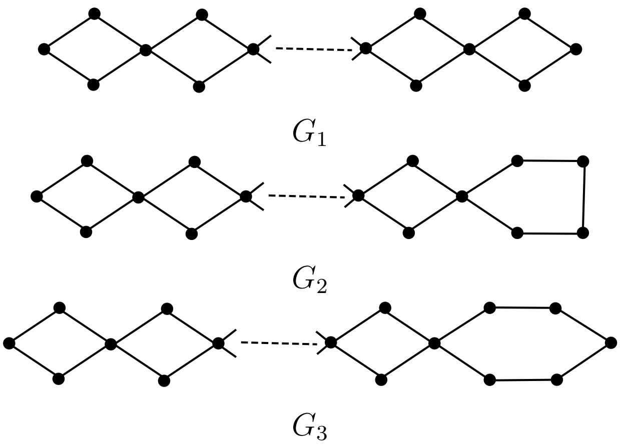

Figure 2. Graphs for the tightness of Theorem 2.2.

The three graphs G_1, G_2, G_3 in Figure 2 are 2-edge-connected. The order of G_i ($i = 1, 2, 3$) is $3k+i$, and $d(G_1) = d(G_2) = 2k$ and $d(G_3) = 2k+1$. From the above result and $d(G) \leq rc(G)$, we have that $rc(G_1) = rc(G_2) = 2k$ and $rc(G_3) = 2k+1$. Hence, the upper bound is tight. $\square$

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